Quantum designs and difference multisets

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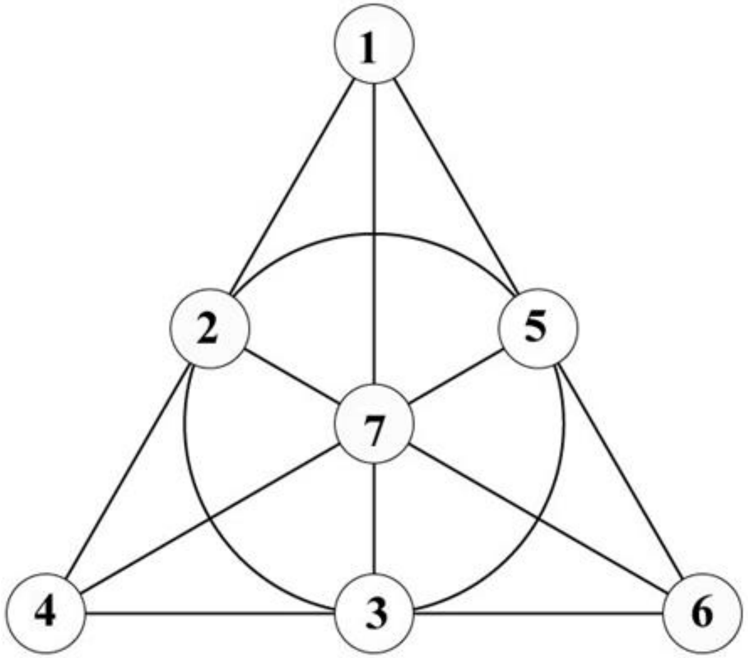
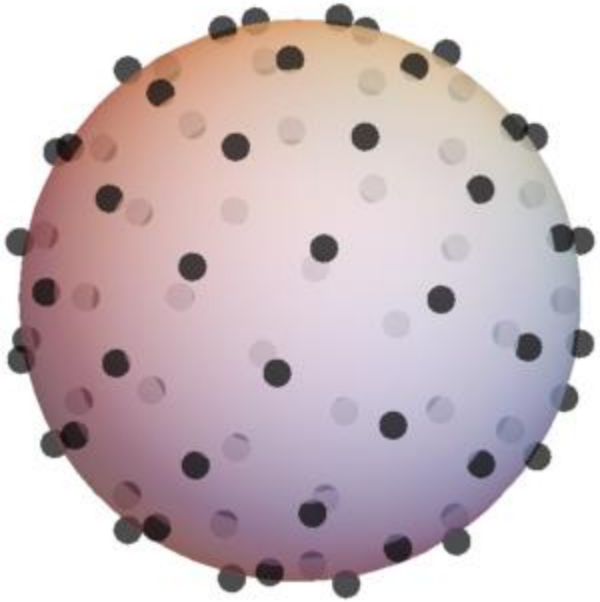
What are those quantum designs?
Let’s start with
Combinatorial design: system of sets having symmetry

1	2		4			
	2	3		5		
		3	4		6	
			4	5		7
1				5	6	
	2				6	7
1		3				7

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9



Quantum designs
Spherical *t*-designs Unitary designs



Quadrature on a sphere
Accurate for polynomials up to order t

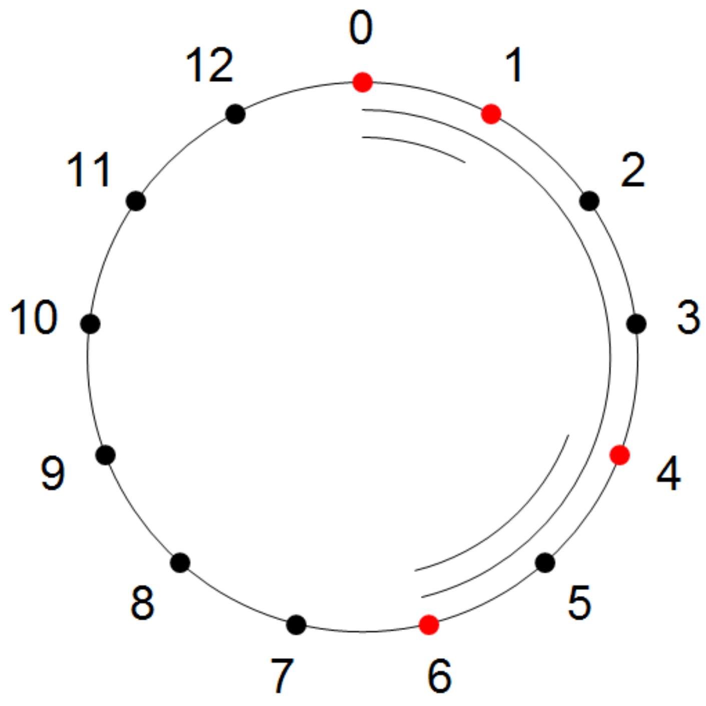
$U(d)$ – group of unitary $d \times d$ matrices
$f(U)$ – homogenous function
If

$$\frac{1}{|X|}\sum_{U\in X} f(U) = \int_{U(d)} f(U)dU$$

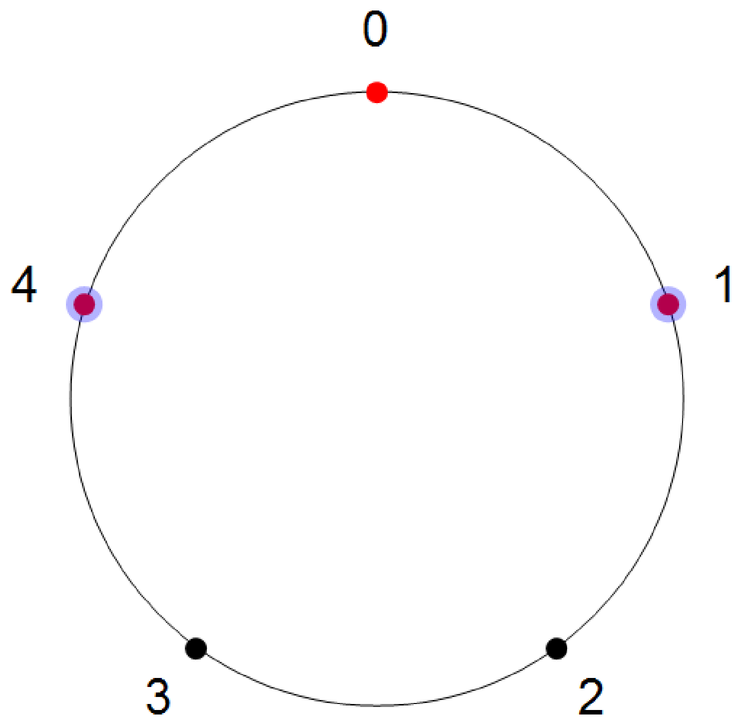
for all $f(U)$ of degree t in U and U^*
then X is a unitary t-design

This is also a quadrature!

A sub-problem: difference multisets



Difference set $\{0,1,4,6\}$ in $\mathbb{Z}_{13}$



Difference multiset $\{0,1,1,4,4\}$ in $\mathbb{Z}_5$

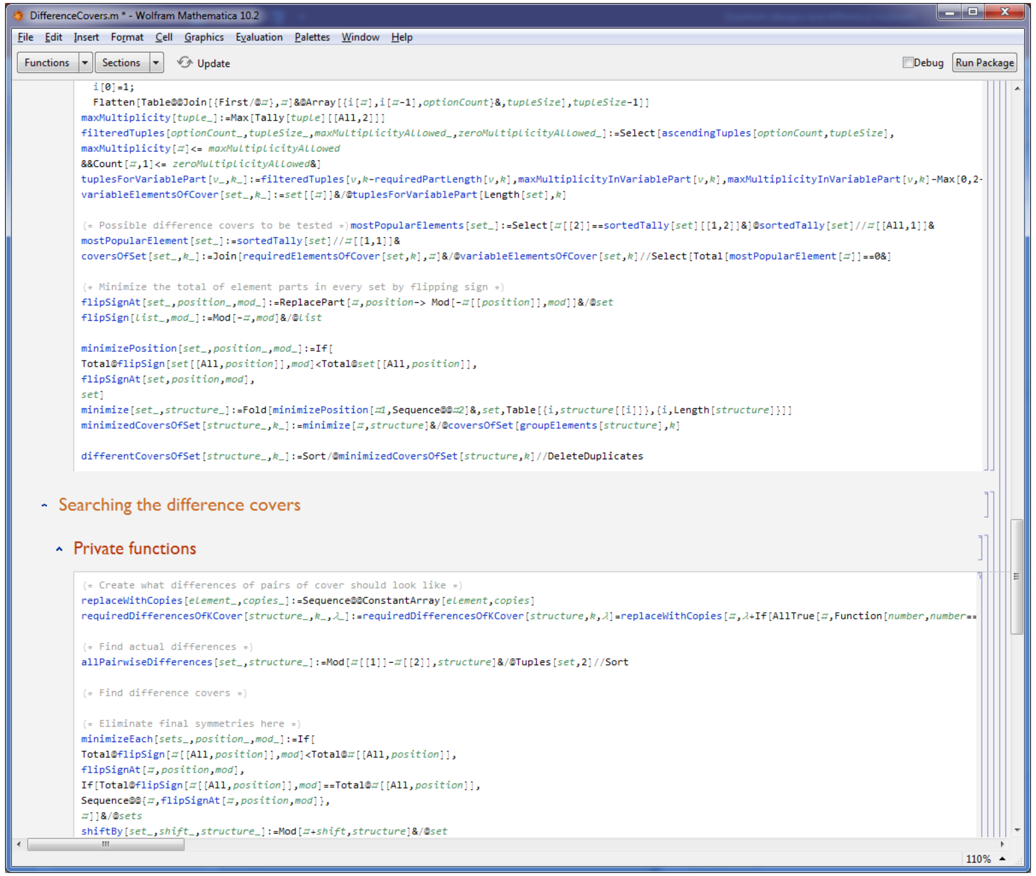
Notation and questions

- (v,k,λ)-difference multiset – a multiset of size k that produces each element of group G ($|G|=v$) exactly λ times as difference of multiset's elements.

$\{0,1,1,4,4\}$ produces each of $v=5$ elements of $\mathbb{Z}_5=\{0,1,2,3,4\}$ exactly $\lambda=4$ times.

- For which v,k,λ do difference multisets exist and when don't they exist?
- What are the constructions of difference multisets?

Step 1: Computer search




```
printDifferenceCoversUpToComplexity[30 000];
{3, 3, 2} → {{0}, {0}, {1}}
{3, 4, 4} → {{0}, {0}, {1}, {1}}
{3, 7, 14} → {{0}, {0}, {0}, {0}, {1}, {1}, {2}}
{6, 4, 2} → {{0, 0}, {0, 0}, {0, 1}, {1, 1}}
{5, 5, 4} → {{0}, {0}, {1}, {1}, {3}}
            {{0}, {0}, {1}, {2}, {2}}
{3, 10, 30} →
{3, 12, 44} → {{0}, {0}, {0}, {0}, {0}, {0}, {1}, {1}, {1},
{5, 6, 6} →
{3, 13, 52} → {{0}, {0}, {0}, {0}, {0}, {0}, {1}, {1}, {1},
{3, 16, 80} → {{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {1},
{4, 9, 18} → {{0}, {0}, {0}, {1}, {1}, {1}, {2}, {2}, {2}}
{3, 19, 114} → {{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}
{3, 21, 140} → {{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}
               {{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}
{3, 22, 154} →
{3, 25, 200} → {{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}
{3, 27, 234} →
 {{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {C
{3, 28, 252} →
 {{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {C
{3, 30, 290} →
```

Computer search results

- For some parameters there are multiple, for some there are none… It's not easy to notice a structure.
- There's a lot of difference multisets over $\mathbb{Z}_3$.

- Let's focus on those $\mathbb{Z}_3$ difference multisets!

Next step: Analytic approach

$$\begin{cases} 3\lambda = k(k-1) \\ \sum n_i = k \\ \sum n_i n_{i+1} = \lambda \end{cases}$$

Theorem 4.1. *Multiplicities of different $(\mathbb{Z}_3, k)$-difference multiset elements i un j are related via*

$$n_{i \neq j} = \frac{k - n_j \pm \sqrt{\frac{4k-(k-3n_j)^2}{3}}}{2} \qquad (8)$$

But for which *k* can we find non-negative integers n_i and n_j that satisfy this?

Improve computer search

$$n_{i \neq j} = \frac{k - n_j \pm \sqrt{\frac{4k-(k-3n_j)^2}{3}}}{2}$$

$$\Rightarrow \quad \frac{k - 2\sqrt{k}}{3} \leq n_j \leq \frac{k + 2\sqrt{k}}{3}$$

3	{0, 1, 2}
4	{0, 2, 2}
7	{1, 2, 4}
9	{1, 4, 4}
9	{2, 2, 5}
….	
4 318 617	{1 438 817, 1 438 876, 1 440 924}
4 318 621	{1 438 156, 1 440 185, 1 440 280}
4 318 639	{1 438 172, 1 440 082, 1 440 385}
4 318 639	{1 438 249, 1 439 774, 1 440 616}

There are difference multisets over $\mathbb{Z}_3$ for k=3,4,7,9,12,13,16,19,21,25,27,28,31,36,…

How are these values special?

Ask, and it will be given to you.
Seek, and you will find.

THE ON–LINE ENCYCLOPEDIA
OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

3,4,7,9,12,13,16,19,21,25,27,28,31,36 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:3,4,7,9,12,13,16,19,21,25,27,28,31,36**

Displaying 1-5 of 5 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A003136 Loeschian numbers: numbers of the form x^2 + xy + y^2; norms of vectors in A2 lattice. +20
(Formerly M2336) 88

0, 1, **3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36**, 37, 39, 43, 48, 49, 52, 57, 61, 63, 64, 67, 73, 75, 76, 79, 81, 84, 91, 93, 97, 100, 103, 108, 109, 111, 112, 117, 121, 124, 127,

Löschian numbers. Is there a link?

Recall: $\frac{k-2\sqrt{k}}{3} \le n_j \le \frac{k+2\sqrt{k}}{3}$

Let's consider multiplicities as «average plus something»: $n_j = \frac{k+\Delta_j}{3}$

$$n_{i\neq j} = \frac{k - n_j \pm \sqrt{\frac{4k-(k-3n_j)^2}{3}}}{2} \Rightarrow n_{i\neq j} = \frac{k - n_j \pm \sqrt{\frac{4k-\Delta_j^2}{3}}}{2}$$

If $k = a^2 + ab + b^2$ then the following values of Δ makes the value under root a perfect square: $\pm(2a+b), \pm(a+2b), \pm(a-b)$

and at least one of them makes whole expression take an integer value.

What about the opposite direction?

Thus *k* being a Löschian number turns out to be enough for this to work.

But are there other difference multisets? Or is k always a Löschian number if a difference multiset exists?

Attempt 1: Check the computer search results.
-- Nope, difference multisets only exist for Löschian k.

But I got stuck and couldn't prove what seemed to be correct.

Ask for help! stackexchange.com

QUESTIONS TAGS USERS BADGES UNANSWERED

certain values seem to make an expression into perfect square, ca

All the variables in this question are in integers. I am trying to prove that

$$\frac{4k - \Delta^2}{3}$$

is a perfect square **only if** $\Delta \in \{\pm(2a+b), \pm(a+2b), \pm(a-b)\}$ where a, b are such that $k = a^2 + ab + b^2$.

1 Answer

active oldest **votes**

2

Suppose $\frac{4k-\Delta^2}{3}$ is a square, in particular say

$$\frac{4k - \Delta^2}{3} = M^2.$$

Then $4k = \Delta^2 + 3M^2$.

Reducing modulo 4, we see that $\Delta^2 + 3M^2 = 0 \pmod 4$, so $\Delta^2 = M^2 \pmod 4$. Hence, $\Delta = M \pmod 2$, so they are both even or both odd.

If they are both odd:

Set

$$a = \frac{\Delta + M}{2}, \quad b = \frac{M - \Delta}{2}$$

which are certainly both integers. It can be checked that $a^2 + ab + b^2 = \frac{1}{4}(\Delta^2 + 3M^2) = k$

Also, $\Delta = a - b$, so Δ can be written as $a - b$, where a, b satisfy $k = a^2 + ab + b^2$

If they are both even:

Similarly, set

$$a = \frac{\Delta - M}{2}, \quad b = M$$

both integers.

Again, we have $a^2 + ab + b^2 = k$, and $\Delta = 2a + b$, as required.

Ultimately, we have shown that for any Δ which makes $\frac{4k-\Delta^2}{3}$ a square number, there are a, b satisfying $a^2 + ab + b^2 = k$ and $\Delta = 2a + b$ or $\Delta = a - b$.

share cite edit flag   edited Nov 26 '17 at 1:51   answered Nov 26 '17 at 1:40

B. Mehta

8,867 14 36

Put it all together, draw conclusions, clean it up... the final result is here!

Theorem 4.5. *For every pair* $a, b \in \mathbb{Z}$ *such that* $k = a^2 + ab + b^2$ *and* $a \geq b \geq 0$ *there are exactly* $-(k+1) \mod 3$ *(up to automorphisms)* $(\mathbb{Z}_3, k)$*-difference multisets and the multiplicities of their elements are*

- $n_i = \frac{k+\Delta_i}{3}$ *for one and* $n_i = \frac{k-\Delta_i}{3}$ *for the other if* $3 \mid k$.
- $n_i = \frac{k+\Delta_i}{3}$ *if* $3 \nmid k$ *un* $b - a \equiv 1 \mod 3$.
- $n_i = \frac{k-\Delta_i}{3}$ *if* $3 \nmid k$ *un* $a - b \equiv 1 \mod 3$.

Thank you for your attention!

Any questions?