Grover's search algorithm with errors

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The unstructured search problem

- Informally: search in the unsorted array

0	0	0	1	0	…	0	0	1	0

- Formaly: we have a function given as a black-box:

$$f(x) : \{1 \ldots N\} \rightarrow \{0,1\}$$

The unstructured search problem is to find $x \in \{1 \ldots N\}$ such that $f(x) = 1$, or to conclude that no such x exists.

Classical case

- Search in the unsorted array

0	0	0	1	0	…	0	0	1	0

- Sequentially check all elements of the array
  - Best case: 1 step
  - Worst case: N steps
  - Average case: N/2 steps

Quantum case: Grover’s algorithm

- Search in the unsorted array

0	0	0	1	0	…	0	0	1	0

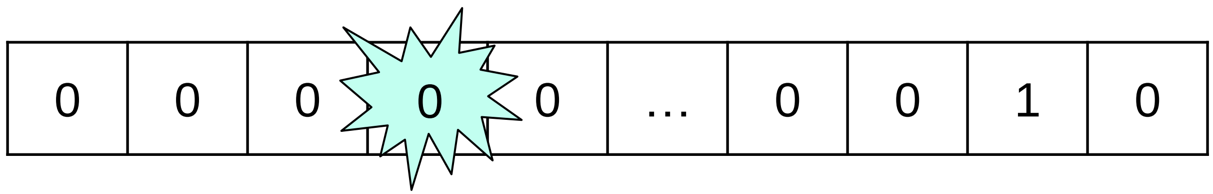
- [Gro96] L. Grover. A fast quantum mechanical algorithm for database search.

  Unstructured search space of N elements can be searched in $O(\sqrt{N})$ steps

- Parallelly check all elements of the array, “merge” results

Research problem

- Suppose a look-up of a value of an element may fail.



- How this affects Grover's search algorithm ?
- Motivation:
  - Study how "data access" errors affects the algorithm
  - Study if Grover's algorithm can be used for data of probabilistic nature (fuzzy search, etc.)

Fuzzy search

- We are given a pattern P and a set of strings { S_1, …, S_N }

P :	1	1	…	0	…	1	0	1	0	Pattern
S_1 :	1	0	…	0	…	0	1	1	0	
S_2 :	0	1	…	1	…	1	0	1	1	
…	…	…	…	…	…	…	…	…	…	Search space
S_N :	0	1	…	1	…	0	1	1	0	

- Find a string S_j which is approximately equal to P.

Fuzzy search in query model

- We are given a pattern P and a set of strings $\{ S_1, \ldots, S_N \}$

P :	1	1	…	0	…	1	0	1	0	Pattern
S_1 :	1	0	…	0	…	0	1	1	0	Search space
S_2 :	0	1	…	1	…	1	0	1	1	
…	…	…	…	…	…	…	…	…	…	
S_N :	0	1	…	1	…	0	1	1	0	

On step t query returns if $P[t] = S_j[t]$

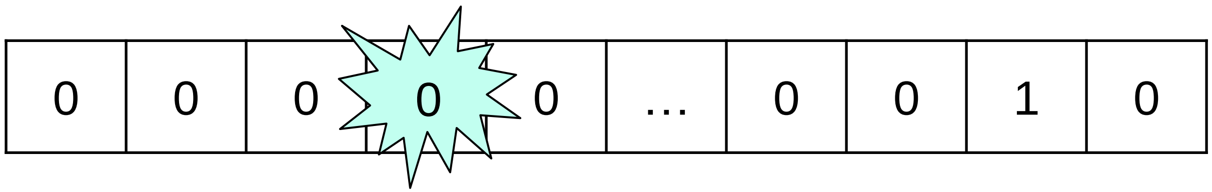
- Find a string S_j which is approximately equal to P.

Error model

- An unstructured search space of N elements
  - $N - k$ elements have value 0 — Non-marked elements
  - k elements $i_1, \dots, i_k$ have value 1 — Marked elements
- Each marked element i_j with probability p_j is reported as non-marked, independent on probabilities of other elements.

Classical case

- Suppose a look-up of a value of an element may fail.

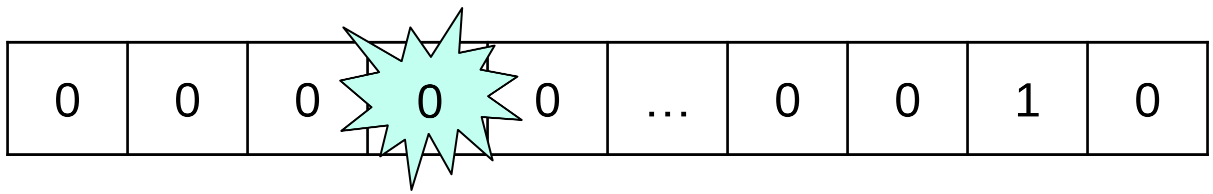


- If the probabilities of look-up errors are known there is simple formula, which gives a probability of finding one of marked elements.

- If there are marked elements with $p_j = 0$ then the algorithm will find one of marked elements.

Quantum case

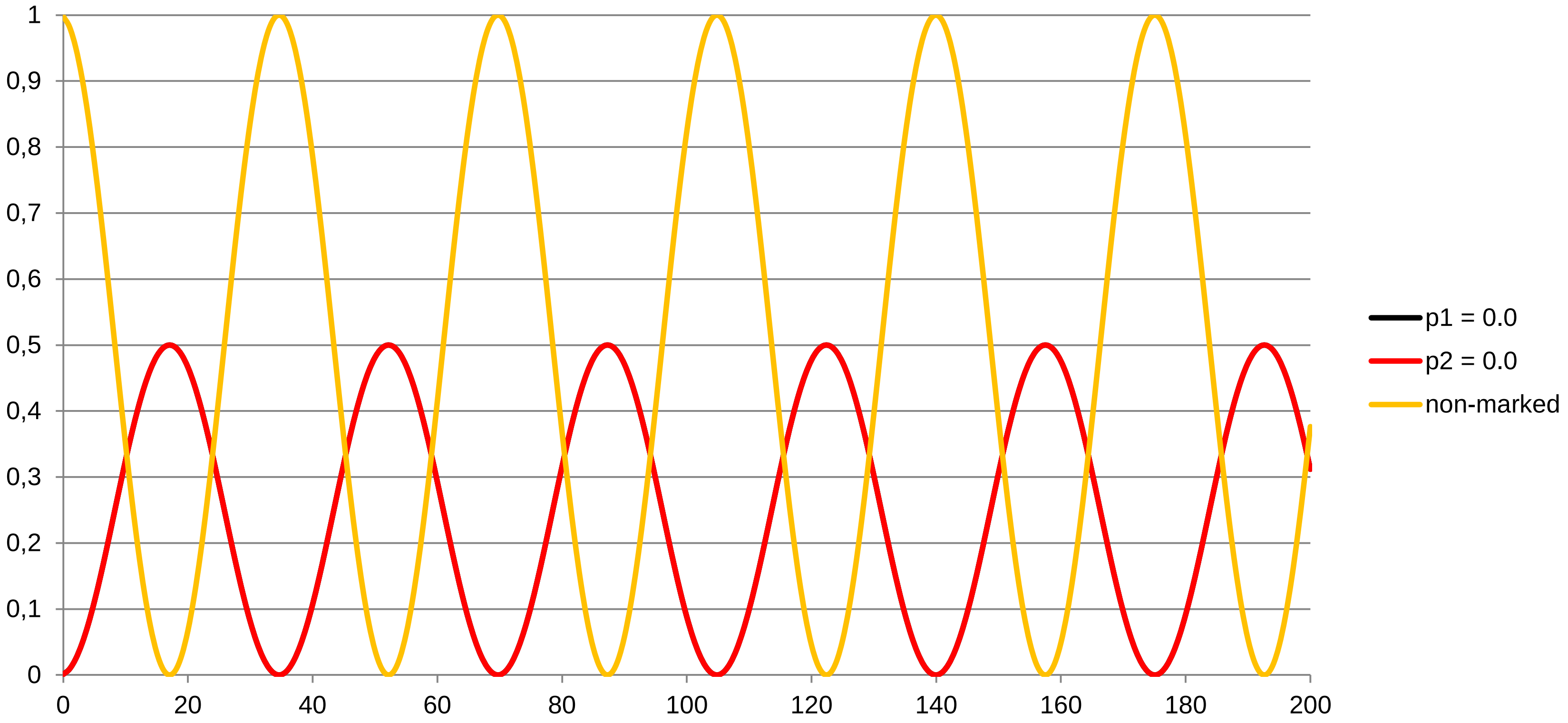
- Suppose a look-up of a value of an element may fail.



- Array elements are checked in parallel.
- The effect of a look-up errors is not trivial.

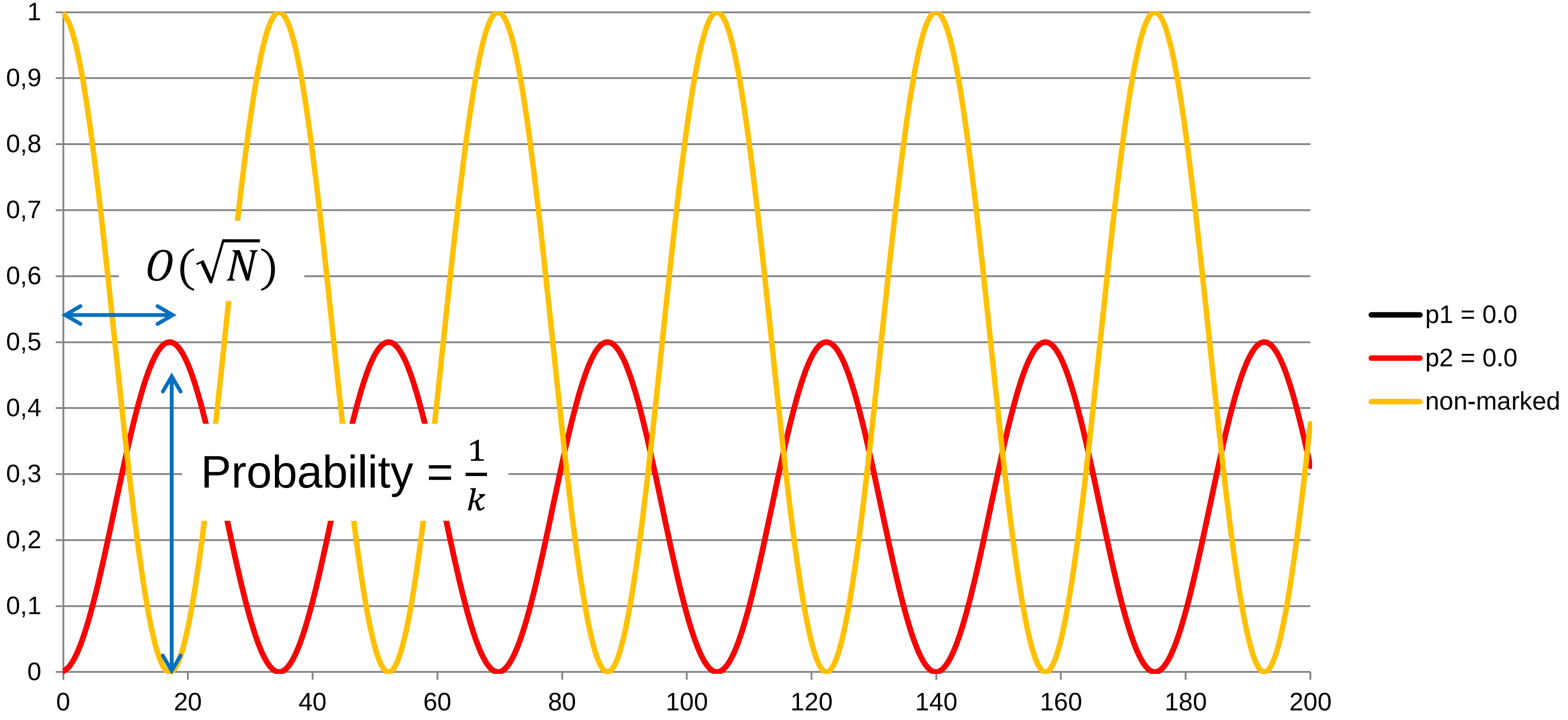
Grover's algorithm without errors

- $N=1000$, $p_1=0$, $p_2=0$



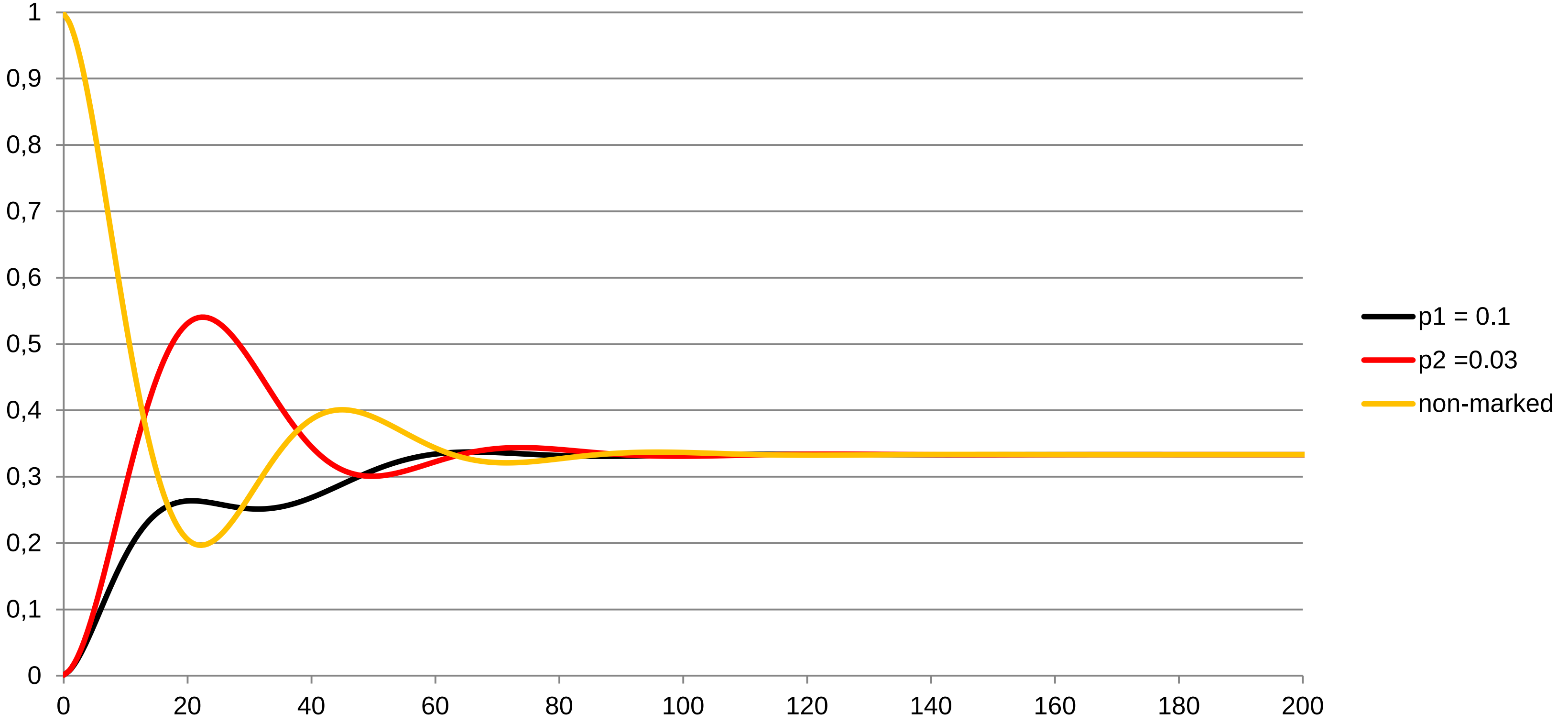
Grover's algorithm without errors

- N=1000, p_1=0, p_2=0



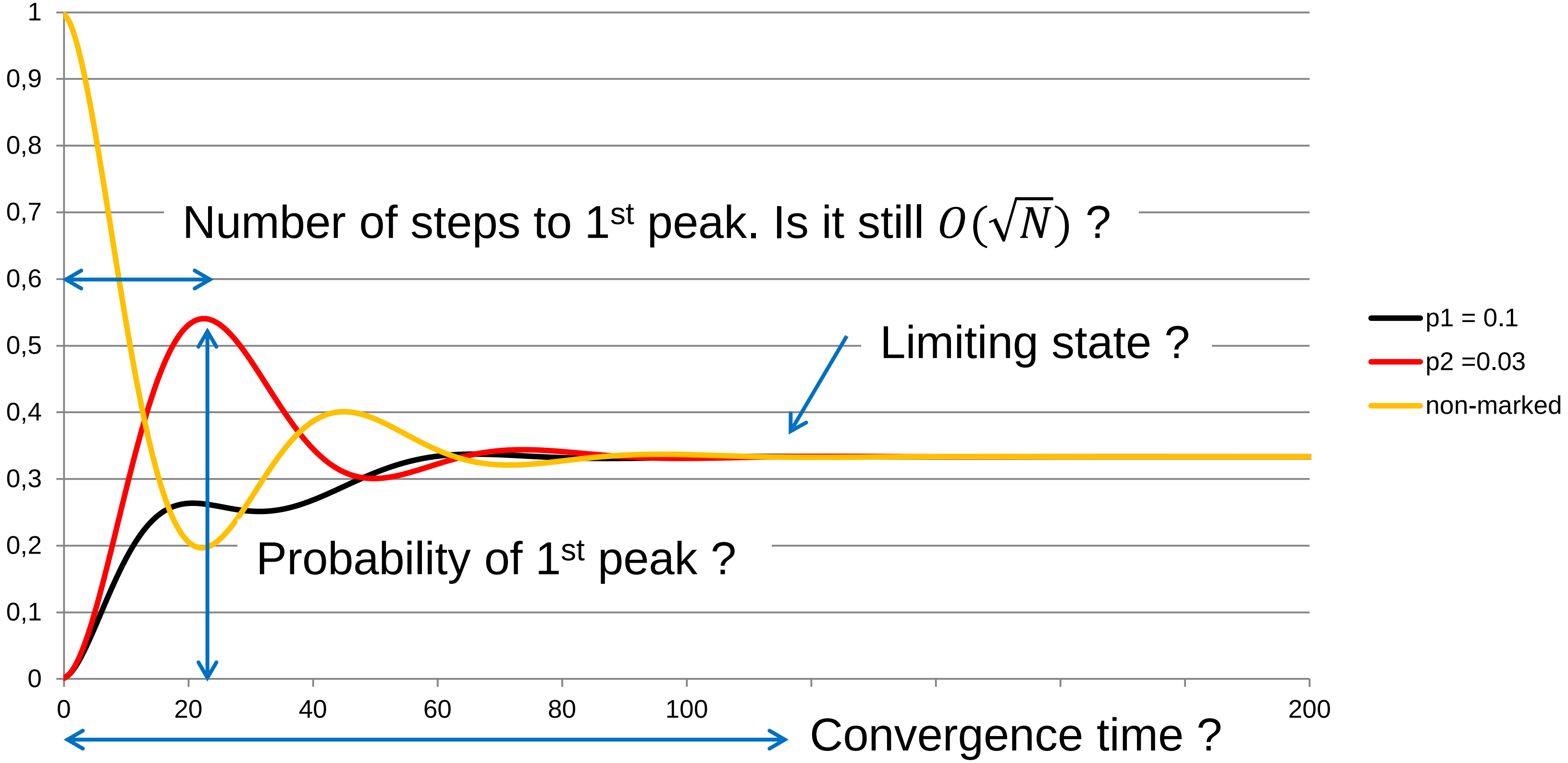
Grover's algorithm with errors

- N=1000, p_1=0.1, p_2=0.03



Grover's algorithm with errors

- $N=1000$, $p_1=0.1$, $p_2=0.03$



Grover's algorithm with errors

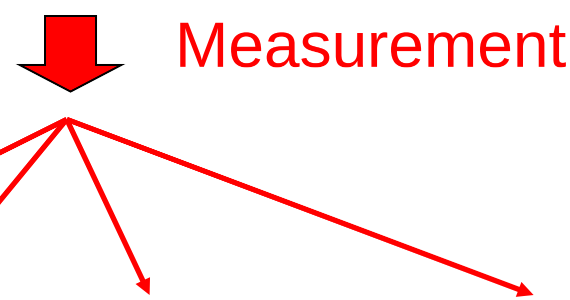
- We study
  - Limiting state
  - Convergence time
  - Probability of 1st peak
  - Number of steps to 1st peak
- We examine two distinct cases
  - All $p_j \neq 0$
  - Some of $p_j = 0$

Are very different in the limit

Limiting state

- If we run Grover's algorithm with faulty query the state of the algorithm converges to

$$\rho_{\lim} = \frac{1}{k+1}\sum_{j=1}^{k} |i_j\rangle\langle i_j| + \frac{1}{k+1}|\psi_-\rangle\langle\psi_-|$$



	i_1	$\cdots$	i_k	non-marked element
Probability	$\frac{1}{k+1}$	$\cdots$	$\frac{1}{k+1}$	$\frac{1}{k+1}$

Limiting state

- If we run Grover's algorithm with faulty query the state of the algorithm converges to

$$\rho_{\lim} = \frac{1}{k+1}\sum_{j=1}^{k} |i_j\rangle\langle i_j| + \frac{1}{k+1}|\psi_-\rangle\langle\psi_-|$$

Measurement

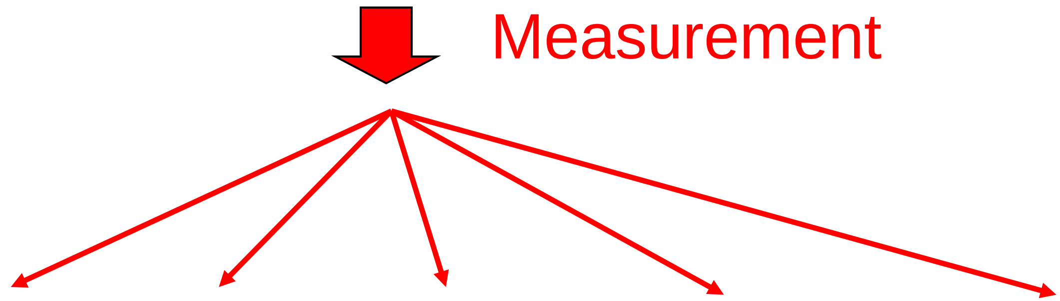
Total probability to measure one of marked elements is $\frac{k}{k+1}$

	i_1	$\cdots$	i_k	non-marked element
Probability	$\frac{1}{k+1}$	$\cdots$	$\frac{1}{k+1}$	$\frac{1}{k+1}$

Limiting state

- If we run Grover's algorithm with faulty query the state of the algorithm converges to

$$\rho_{\lim} = \frac{1}{k+2}\sum_{j=1}^{k}|i_j\rangle\langle i_j| + \frac{1}{k+2}|\psi_+\rangle\langle\psi_+| + \frac{1}{k+2}|\psi_-\rangle\langle\psi_-|$$

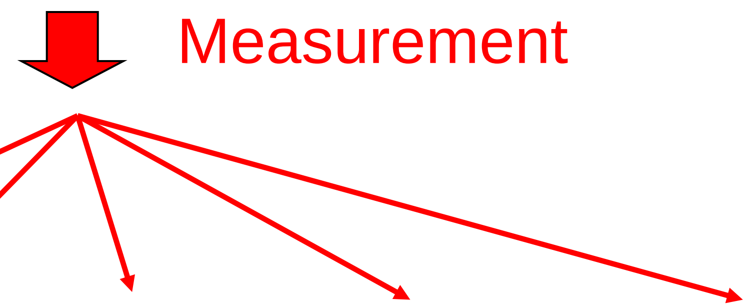


	i_1	$\cdots$	i_k	marked element	non-marked element
Probability	$\frac{1}{k+2}$	$\cdots$	$\frac{1}{k+2}$	$\frac{1}{k+2}$	$\frac{1}{k+2}$

Limiting state

- If we run Grover's algorithm with faulty query the state of the algorithm converges to

$$\rho_{\lim} = \frac{1}{k+2}\sum_{j=1}^{k}|i_j\rangle\langle i_j| + \frac{1}{k+2}|\psi_+\rangle\langle\psi_+| + \frac{1}{k+2}|\psi_-\rangle\langle\psi_-|$$



Total probability to measure one of marked elements is $\frac{k+1}{k+2}$

	i_1	$\cdots$	i_k	marked element	non-marked element
Probability	$\frac{1}{k+2}$	$\cdots$	$\frac{1}{k+2}$	$\frac{1}{k+2}$	$\frac{1}{k+2}$

Limiting state

- All $p_j \neq 0$

$$\rho_{\lim} = \frac{1}{k+1}\sum_{j=1}^{k} |i_j\rangle\langle i_j| + \frac{1}{k+1}|\psi_-\rangle\langle\psi_-|$$

Probability to measure one of marked elements is $\frac{k}{k+1}$

- Some of $p_j = 0$

$$\rho_{\lim} = \frac{1}{k+2}\sum_{j=1}^{k} |i_j\rangle\langle i_j| + \frac{1}{k+2}|\psi_+\rangle\langle\psi_+| + \frac{1}{k+2}|\psi_-\rangle\langle\psi_-|$$

Probability to measure one of marked elements is $\frac{k+1}{k+2}$

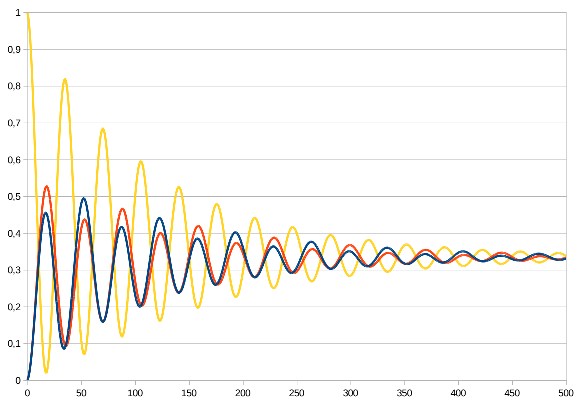
Convergence time

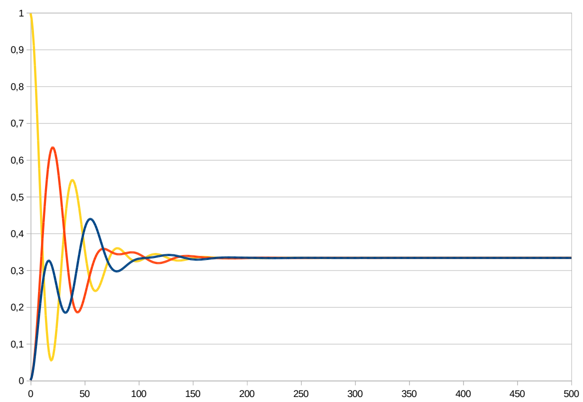
- Convergence time is $O(N)$.

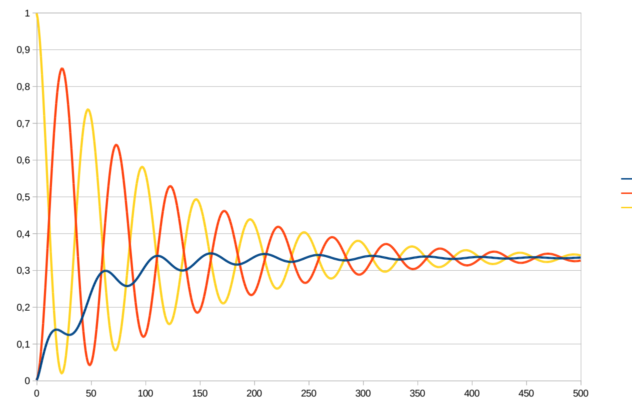
For every $\epsilon > 0$ there exists a number of steps of the algorithm $t = O(N)$, for which the probability to find one of the marked elements is in $[\frac{k}{k+1} - \epsilon, \frac{k}{k+1} + \epsilon]$.

Convergence time

- Convergence time is $O(N)$.
- Convergence time is highly dependent on probability of error of marked elements.







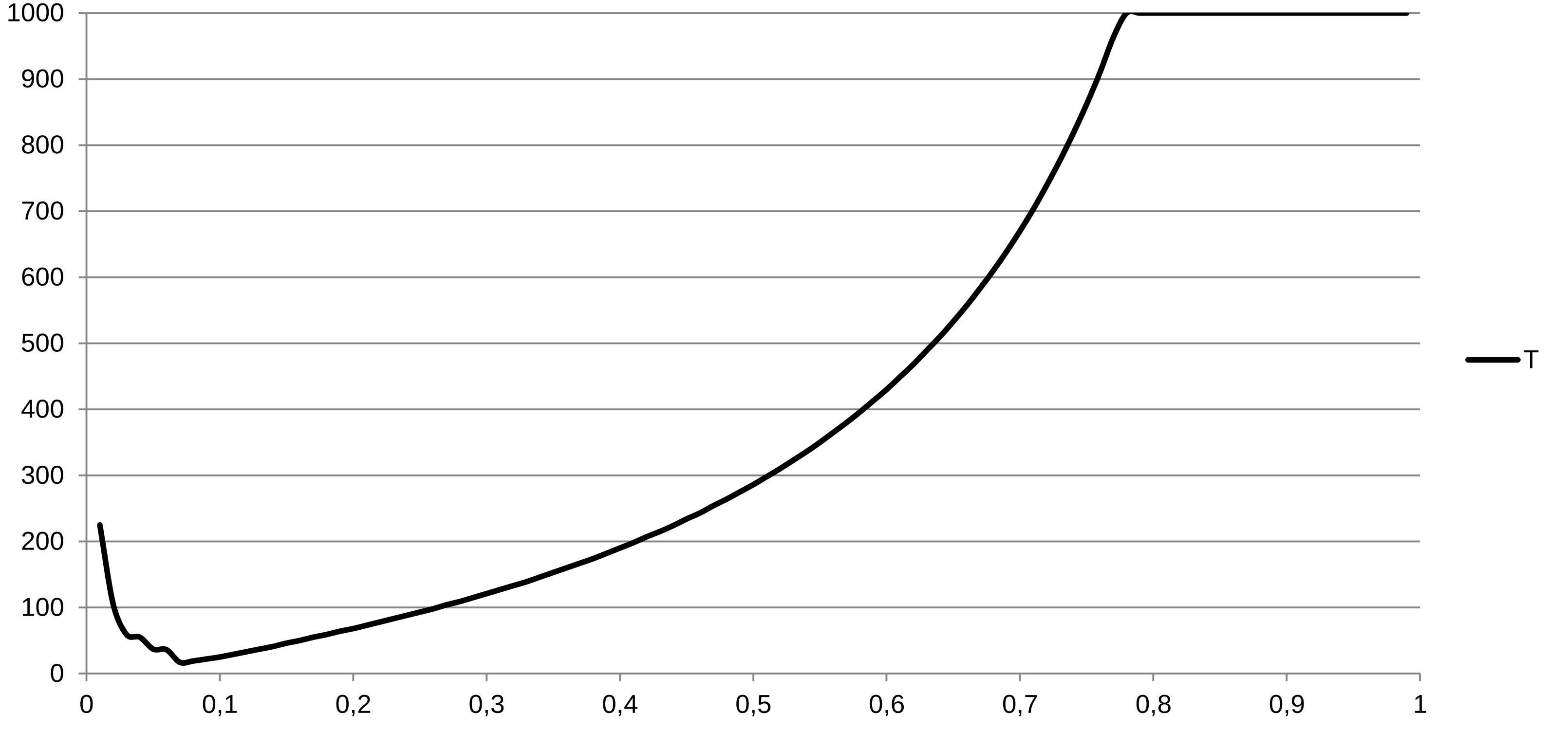
$N=1000,\ p_1=0.01,\ p_2=0$ $N=1000,\ p_1=0.05,\ p_2=0$ $N=1000,\ p_1=0.2,\ p_2=0$

Convergence time

- Convergence time is $O(N)$.
- Convergence time is highly dependent on probability of error of marked element.
- Convergence time in not linear on probability of error of marked element.

Convergence time

- N=1000



First peak

- All $p_j \neq 0$

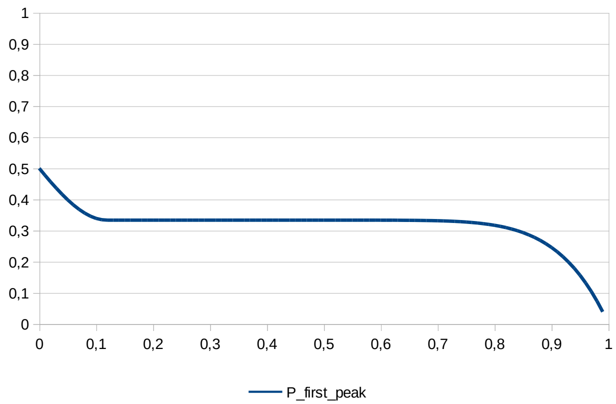
  No speed-up over classical exhaustive search is possible

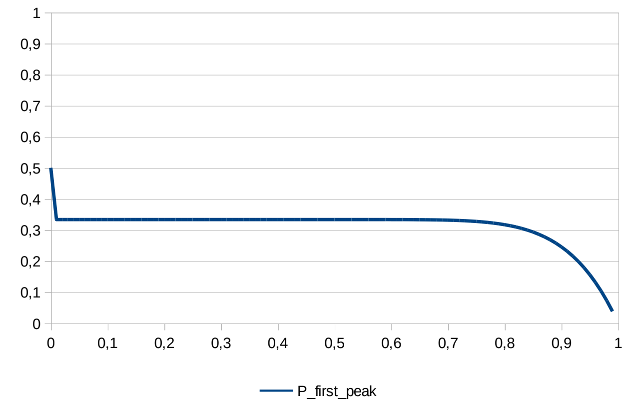
- Some of $p_j = 0$

  “Converges” to Grover’s search with $\{p_j : p_j = 0\}$ marked elements

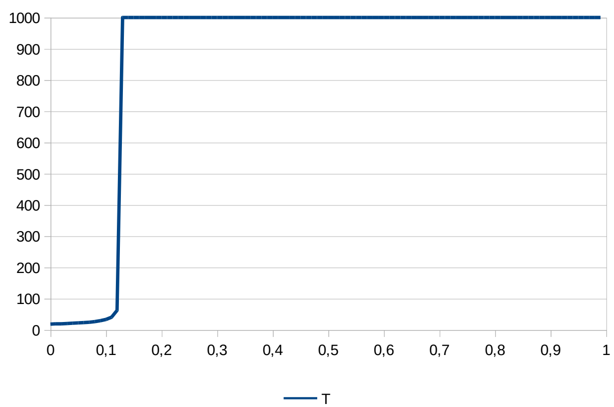
First peak: some $p_j \neq 0$

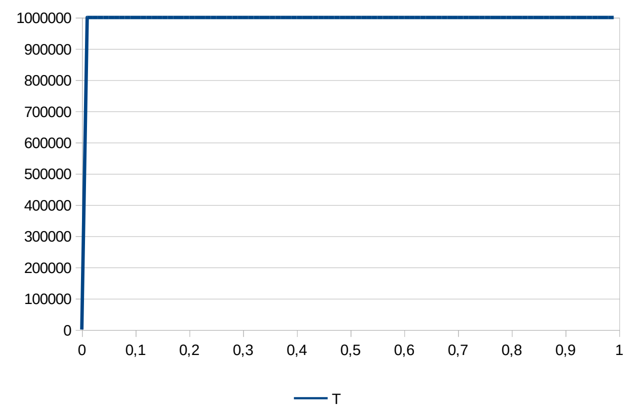
Probability of 1st peak





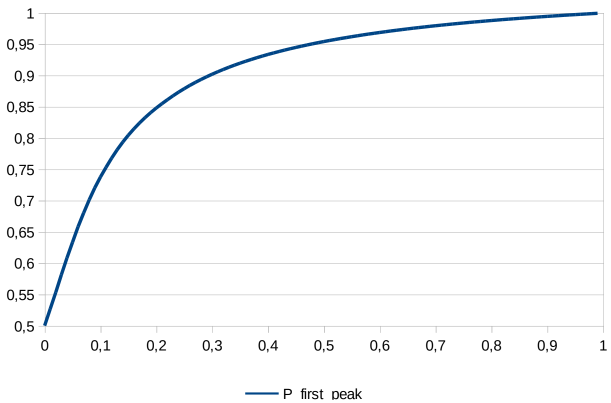
Number of steps



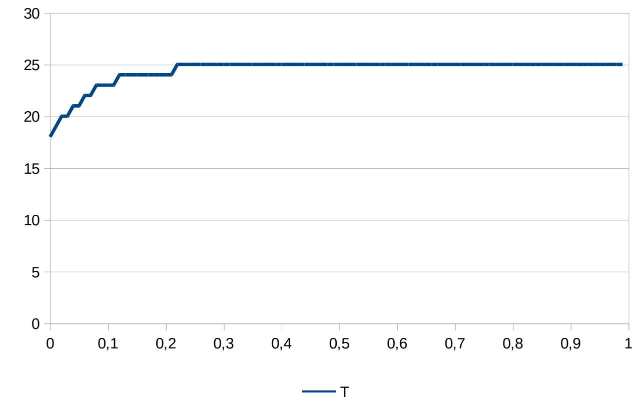


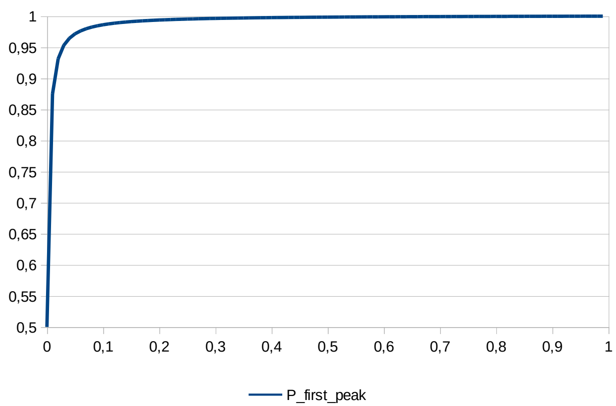
N = 1,000 → N = 1,000,000

First peak: some $p_j = 0$

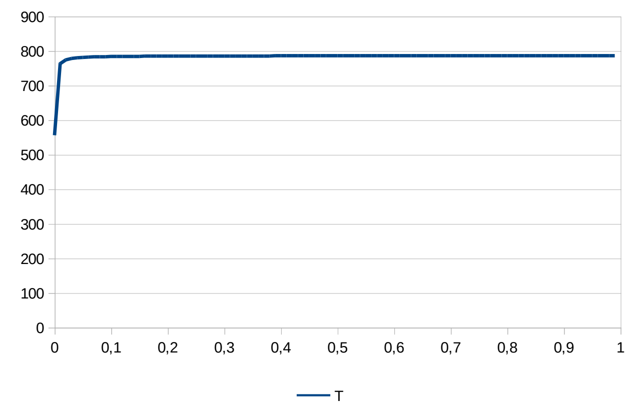


Probability of 1st peak





Number of steps



N = 1,000 → N = 1,000,000

Things to be done

- More precise estimate of convergence time.
- Analytical formula for maximal probability to find one of marked elements.
- Algorithmic applications

Algorithmic applications

- We are given a set of "almost" equal strings $\{S_1, \ldots, S_N\}$
- Find S_i which differs from other S_j

S_1:	0	1	…	1	…	0	1	1	0
S_2:	0	1	…	1	…	0	0	1	0
…	…	…	…	…	…	…	…	…	…
S_N:	0	1	…	1	…	0	1	1	0

On step t query returns $S_j[t]$

Thank you !

Bibliography

- [Gro96] L. K. Grover.

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